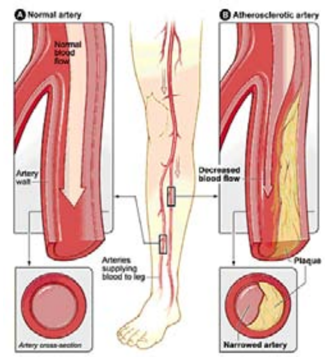
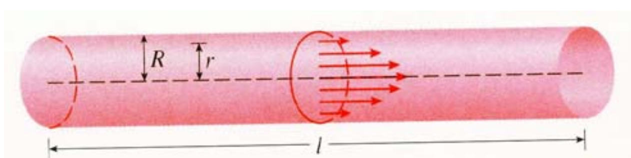
Calculus I Project: Rates of Change

Introduction: The goal of this project is to use differential calculus and computational thinking to solve engineering problems. There are many areas of engineering that involve flow of a liquid or gas through a 'tube'. Blood flows through a vessel (biomedical), water through a pipe (civil), electricity through a current (electrical), and waste through a sewer system (chemical, civil, bio-systems). This project focuses on the flow of blood in a blood vessel. As a result the students will see how differential calculus and computational thinking are applied to model and solve real world problems and, in turn, gain a better understanding of the vascular system. Moreover, students will have a chance to explore the use of computational software such as Maple as a tool for mathematical computations and visualization.



Background: The flow of blood through a blood vessel can be modeled using a cylindrical tube of radius R and length l to represent the blood vessel. Due to friction, the velocity v of the blood is greatest along the central axis of the tube and decreases the closer (r) the blood is to the walls of the tube. The relationship between v and r is given by the Law of Laminar Flow discovered by Jean-Louis-Marie Poiseuille in 1840. This law states that:



$$v = \frac{P}{4\eta l}\left(R^2 - r^2\right)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constants, then v is a function of r with domain $[0, R]$.

Example: Blood Flow. Using the formula $v = \frac{P}{4\eta l}\left(R^2 - r^2\right)$

1. Evaluate the rate of change of v with respect to r which is called the **velocity gradient.**
2. Calculate the velocity and velocity gradient when

   $\eta = 0.027, R = 0.008$ cm, $l = 2$ cm, $P = 4000$ dynes/cm^2 and $r = 0.002$ cm

3. Graph the velocity and velocity gradient functions when
   $\eta = 0.027, R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2

Solution: A solution using Maple follows.

Example: Blood Flow

The velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the
wall. The relationship between v and r is given by the Law of Laminar Flow discovered by French physician Jean-Louis-Marie Poiseuille in 1840.

$$v := \frac{P}{4\,\eta\cdot l}\left(R^2 - r^2\right);$$

$$\frac{1}{4}\,\frac{P\left(R^2 - r^2\right)}{\eta\, l} \tag{1}$$

where η is the viscosity of the blood and P the pressure difference between the ends of the tube. If P and l are constant, v is a function of r with domain [0,R].

1. Average vs instantaneous rate of change

The <u>average</u> rate of change of the velocity from $r = r_1$ to $r = r_2$ (moving outward to the wall) is given by:

$$\frac{\Delta v}{\Delta r} = \frac{\left(v(r2) - v(r1)\right)}{r2 - r1}$$

$$\frac{\Delta v}{\Delta r} = \frac{v(r2) - v(r1)}{r2 - r1} \tag{1.1}$$

When $\Delta r \rightarrow 0$, the <u>instantaneous</u> rate of change of the velocity is obtained. This is called the **velocity gradient**.

$$\textbf{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

$$\lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr} \tag{1.2}$$

Therefore an expression for the velocity gradient can be found by taking the derivative of (1) above.

$$\frac{d}{d\,r}\ \textbf{(1)}$$

$$-\frac{1}{2}\,\frac{P\,r}{\eta\, l} \tag{1.3}$$

Hence the **velocity gradient** is $-\frac{1}{2}\,\frac{P\,r}{\eta\, l}$

2. Example using a smaller human artery: η = 0.027, R = 0.008 cm, l = 2 cm, P = 4000 dynes/cm^2

$\eta = 0.027$

$$\eta = 0.027 \quad (1.1.1)$$

$R = 0.008$

$$R = 0.008 \quad (1.1.2)$$

$l = 2$

$$l = 2 \quad (1.1.3)$$

$P = 4000$

$$P = 4000 \quad (1.1.4)$$

The velocity of the blood is:

$\frac{P}{4 \cdot \eta \cdot l} \cdot (R^2 - r^2)$

$$\frac{1}{4} \frac{P (R^2 - r^2)}{\eta l} \quad (1.1.5)$$

$\xrightarrow{\text{evaluate at point}}$

$$1.185185185 - 18518.51851 r^2$$

At a specific value for r: r = .002 cm

$1.185185185 - 18518.51851 r^2$

$$1.185185185 - 18518.51851 r^2 \quad (1.1.6)$$

$\xrightarrow{\text{evaluate at point}}$

$$1.111111111$$

So the velocity of the blood .002 cm away from the center axis of the blood vessel is 1.11 cm/s.

To find the value of the velocity gradient (instantaneous rate of change of the flow of blood at r = .002 cm) substitute the values into **(1.3)**

$-\frac{1}{2} \frac{P r}{\eta l}$

$$-\frac{1}{2} \frac{P r}{\eta l} \quad (1.1.7)$$

$\xrightarrow{\text{evaluate at point}}$

$$-74.07407410 \quad (1.1.8)$$

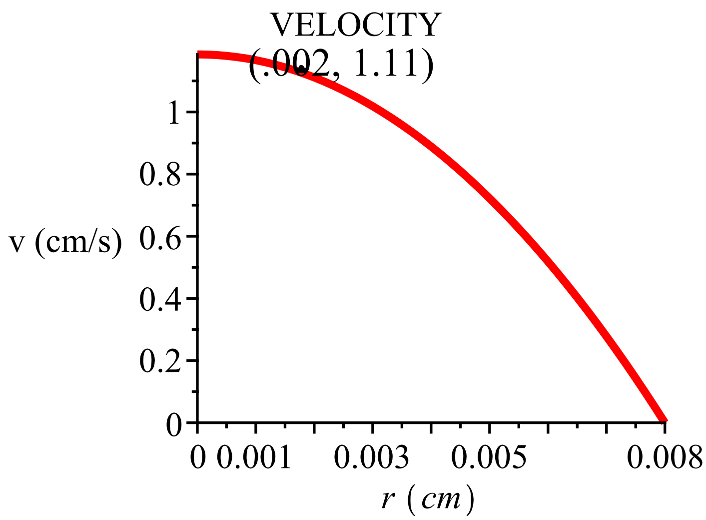
The velocity gradient has a value of -74 (cm/s)/cm

3. Plots of velocity and velocity gradient as functions of r

Velocity Function
(1.1.6)

$$1.185185185 - 18518.51851\, r^2 \qquad \textbf{(1.2.1)}$$

→



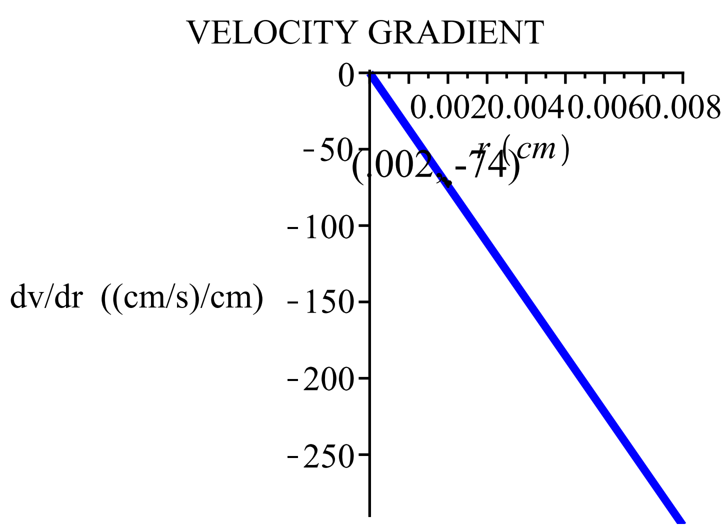
Velocity Gradient
(1.1.7)

$$-\frac{1}{2}\,\frac{P\,r}{\eta\,l} \qquad \textbf{(1.2.2)}$$

evaluate at point →

$$-37037.03702\, r \qquad \textbf{(1.2.3)}$$

→



Assignments:

1. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Calculus II.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

a) Show that the relative change in F is about four times the relative change in R.

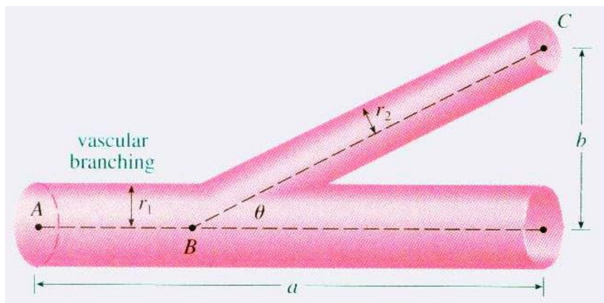
Hint: relative change of $f(x)$ is $\frac{\Delta f}{f_0}$ or $\frac{\Delta y}{y_0}$ and we can estimate the relative change using differentials: $\frac{\Delta f}{f_0} \approx \frac{df}{f_0}$ or $\frac{\Delta y}{y_0} \approx \frac{dy}{y_0}$

b) How will a 5% increase in the radius affect the flow of blood?

2. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expanded by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from the equation: $F = \frac{\pi P R^4}{8\eta l}$ which you will study in Calculus II.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Use Maple to plot two or three graphs of the resistance R as a function of θ for at least two different sets of data. For instance: let $a = 10,\ b = 5, C = 1,\ r_1 = 1, \text{ and } r_2 = 2/3$ be the first data set. Hint: you need to adjust the domain and range so you can see reasonable pictures. From the graphs you plot, what do you observe? Is there a minimum value for R? Where does it occur?

(c) Prove that the resistance function in part (a) is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(d) Find the optimal branching angel (correct to nearest degree) when radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

3. Write at least one page report on what you have learned about computational thinking from doing this project. In your report, you can include a list of mathematical concepts you used, the knowledge you gained about blood flow and the vascular system, the challenges you experienced in connecting math with real world problems. The following survey questions may provide you some ideas in writing your report.

4. Please fill out the project survey form on the next page.

Math 151 Project Survey

		Strongly Agree	Agree	Neutral	Disagree	Strongly Disagree
Numbers	Survey Questions	5	4	3	2	1
1	I understand the importance of domain, range, units, and ratios in problem solving.					
2	I understand the importance of having a logical order in problem solving process.					
3	I have a better understanding of modeling process.					
4	I have a better understanding of computational thinking.					
5	Projects like this help me connect math to the real world.					
6	Projects like this are worthwhile.					

Comments or feedback: